Stochastic resonance of collective variables in finite sets of interacting identical subsystems

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We explore stochastic resonance effects in the response of a complex stochastic system formed by a finite number of interacting, identical subunits driven by a time-periodic force. The driving force alone cannot induce sustained oscillations between the different attractors of the dynamics in the absence of noise. We focus on a global stochastic variable defined as the arithmetic mean of the relevant stochastic variable of each subunit. We construct numerical approximations to its first two long time cumulant moments and its long time correlation function. We also compute the output signal-to-noise ratio and the stochastic resonance gain, for a wide range of parameter values and several types of driving forces. The coupling between the subsystems leads, within adequate ranges of the parameter values, to global outputs with very large signal-to-noise ratios. We have also observed gains larger than unity in the global response to subthreshold sinusoidal driving forces.

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I. INTRODUCTION

Stochastic resonance (SR) is one of several interesting phenomena that have been studied in relation to the response of stochastic nonlinear systems driven by externally controlled, deterministic, time-periodic forces. Several quantities have been introduced to characterize SR, and different theoretical approaches have been put forward to describe them. The theoretical expressions have been tested against extensive numerical calculations as well as experiments and analogous simulations [1]. Numerics has also provided useful insights into the characteristics of the phenomenon in regions of parameter values beyond the validity of analytical approximations [2].

A large proportion of the published work has concentrated on the phenomenon of SR in simple systems [1]. Nonetheless, SR has also been studied in complex systems [3,4]. In [5], a set of globally coupled, noisy bistable systems driven by an external periodic force is described in terms of a nonlinear master equation. Bulsara and collaborators have considered arrays of globally [6] or locally [7] coupled nonlinear oscillators. They find that the stochastic resonant response of one of the oscillators is enhanced as a consequence of its coupling to other identical oscillators. Neiman *et al.* [8] use linear response theory to analyze the system size dependence of the system response, for sets of independent subsystems driven by periodic or aperiodic forces. For a particular kind of coupling (mean-field coupling) [9], we have observed a large enhancement in stochastic resonant effects in arrays of noisy, driven bistable systems in the limit of infinitely large systems [10]. SR in ion channel assemblies have been studied by Schmid *et al.* [11]. More recently, globally coupled networks of noisy neural elements have been studied [12].

In this work, we consider a complex system formed by a finite number of N identical, coupled subunits. Each subunit is characterized by a stochastic variable, say $x_i(t)$, and it is subject to a time-periodic driving term. The consideration of finite sets is inspired by the fact that certain processes in neuroscience seem to involve a rather small number of subsystems [13]. The whole system will be characterized by a single stochastic collective variable $S(t)=(1/N)\Sigma_{i=1}^{N}x_i(t)$. The response of the system to the external driving depends on the peculiarities of the external driving, on the parameter values of the noise terms, on the number of subunits, and on the strength of the coupling terms. We will focus our analysis on the long time limit of the first two cumulant moments of $S(t)$ and on an appropriately defined one-time correlation function. We will see that the first moment will be periodic in time, with the same period as the driving force, and its shape will, in general, be distorted with respect to the shape of the driver. Its amplitude presents a nonmonotonic dependence on the noise strength, typical of a SR phenomenon. In general, the amplitude of the first moment of the collective variable is enhanced with respect to the amplitude of the first moment of $x_i(t)$ in the absence of coupling. The degree of enhancement depends on the system size and the coupling strength. On the other hand, the second cumulant of $S(t)$ is much smaller than the second cumulant of $x_i(t)$. For uncoupled sets, this follows from the central limit theorem [14]. We will see that it is still true in the presence of coupling. This strong reduction of the fluctuations of $S(t)$ in sets of identical, coupled subunits manifests itself on the signal-to-noise ratio of the global output. In general, the output signal $S(t)$ will be much less noisy than the output of an individual, uncoupled subunit. Some years ago, Loerincz, Gingl, and Kiss [15] demonstrated that stochastic resonance gains larger than unity for a single threshold detector subject to strong spiking signals are possible. In this work, we have also obtained values of the stochastic resonance gain larger than unity, for coupled sets driven by subthreshold sinusoidal forces. Even though there are no theoretical reasons why the gain cannot be larger than unity in nonlinear stochastic resonance, we think that, to the best of our knowledge, gains larger than unity have not been previously reported for subthreshold, sinusoidal driving forces.

The rest of the paper is as follows. In the next section, we specify the model system and define the quantities to be

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evaluated: the first two cumulant moments of the collective stochastic variable, $S(t)$ and its one-time correlation function. For uncoupled sets we obtain formal expressions relating those quantities to that of a single unit. When coupling is present, the single unit variables are correlated and the formal expressions need to be modified. Unfortunately, there are no reliable analytical procedures to analyze the formal expressions. Thus, we will rely on numerical procedures to obtain information about the system behavior. The results are presented in Sec. III. Finally, in the last section, we summarize our findings.

II. THE MODEL AND SOME DEFINITIONS

We consider a set of N identical subsystems, each of them characterized by a variable $x_i(t)$ $(i=1,\ldots,N)$ satisfying a stochastic evolution equation (in dimensionless form) of the type

$$\dot{x}_i = -\frac{\partial U(x_1,\ldots,x_N)}{\partial x_i} + F(t) + \xi_i(t). \tag{1}$$

The external driving force is periodic in t, $F(t)=F(t+T)$. The term $\xi_i(t)$ represents a white noise with zero average and $\langle \xi_i(t)\xi_j(s)\rangle = 2D\delta_{ij}\delta(t-s)$. When U is separable, $U(x_1,\ldots,x_N)=\Sigma U_i(x_i)$, the subunits are statistically independent.

The system as a whole will be characterized by a collective variable $S(t)$ defined by

$$S(t) = \frac{1}{N}\sum_j x_j(t). \tag{2}$$

We are interested in the long time response of the collective variable to the driving force, when the system size N is kept finite. In particular, we will concentrate on the analysis of its first two cumulant moments $\langle S(t)\rangle_*$, $\langle S^2(t)\rangle_* - \langle S(t)\rangle_*^2$, and its one-time correlation function, defined as

$$L(\tau) = \frac{1}{T}\int_0^T dt \langle S(t)S(t+\tau)\rangle_*, \tag{3}$$

where the notation $\langle \cdots \rangle$ indicates an average over the noise realizations and the subscript * indicates the long time limit of the noise average, i.e., its value after waiting for t large enough that transients have died out.

Let us first consider the case of *independent subsystems*. It is straightforward to see that

$$\langle S(t)\rangle = \frac{1}{N}\sum_j \langle x_j(t)\rangle = \langle x_1(t)\rangle, \tag{4}$$

as all the individual averages are identical. Also,

$$\langle S^2(t)\rangle = \frac{1}{N^2}\sum_{i,j}\langle x_i(t)x_j(t)\rangle = \frac{1}{N}\langle x_1^2(t)\rangle + \left(\frac{N-1}{N}\right)\langle x_1(t)x_2(t)\rangle. \tag{5}$$

As the subsystems are identical and independent, we have $\langle x_1(t)x_2(t)\rangle = \langle x_1(t)\rangle\langle x_2(t)\rangle = \langle x_1(t)\rangle^2$. Thus,

$$\langle S^2(t)\rangle - \langle S(t)\rangle^2 = \frac{1}{N}[\langle x_1^2(t)\rangle - \langle x_1(t)\rangle^2]. \tag{6}$$

The second cumulant of the collective variable is $1/N$ times that of the individual subsystems, in agreement with the central limit theorem.

The two-time correlation function is

$$\begin{aligned}\langle S(t)S(t+\tau)\rangle &= \frac{1}{N^2}\sum_{i,j}\langle x_i(t)x_j(t+\tau)\rangle \\ &= \frac{1}{N}\langle x_1(t)x_1(t+\tau)\rangle \\ &\quad + \left(1-\frac{1}{N}\right)\langle x_1(t)\rangle\langle x_1(t+\tau)\rangle,\end{aligned} \tag{7}$$

where we have used that the subunits are identical and statistically independent. In the long t limit, the two-time correlation function is periodic in t with the period of the driving force. Then, averaging the long time limit of Eq. (7) over a period, one finds the one-time correlation function

$$\begin{aligned}L(\tau) &= \frac{1}{T}\int_0^T dt\langle S(t)S(t+\tau)\rangle_* = \frac{1}{N}C(\tau) + C_{coh}(\tau)\left(1-\frac{1}{N}\right) \\ &= C_{coh}(\tau) + \frac{1}{N}C_{incoh}(\tau).\end{aligned} \tag{8}$$

Here we have used the individual subsystem correlation function

$$C(\tau) = \frac{1}{T}\int_0^T dt\langle x_1(t)x_1(t+\tau)\rangle_*.$$

It contains two terms: its coherent part $C_{coh}(\tau) = (1/T)\int_0^T dt\langle x_1(t)\rangle_*\langle x_1(t+\tau)\rangle_*$, and its incoherent part $C_{incoh}(\tau) = C(\tau) - C_{coh}(\tau)$. The coherent part is periodic in τ with the period of the driver, while the incoherent part decays to zero as τ goes to ∞. Consequently, $L(\tau)$ is

$$L(\tau) = L_{coh}(\tau) + L_{incoh}(\tau), \tag{9}$$

with

$$L_{coh}(\tau) = C_{coh}(\tau), \quad L_{incoh}(\tau) = \frac{1}{N}C_{incoh}(\tau). \tag{10}$$

The coherent part of the collective variable correlation function, $L_{coh}(\tau)$, is identical to the coherent part of one of the individual subsystems, $C_{coh}(\tau)$, while the corresponding incoherent part is reduced by a factor $1/N$ with respect to $C_{incoh}(\tau)$.

The output signal-to-noise ratio R_{out} is

$$R_{out} = \lim_{\epsilon\to 0^+}\frac{\int_{\Omega-\epsilon}^{\Omega+\epsilon} d\omega \tilde{L}(\omega)}{\tilde{L}_{incoh}(\Omega)} = \frac{\tilde{L}_{coh}(\Omega)}{\tilde{L}_{incoh}(\Omega)}, \tag{11}$$

where Ω is the fundamental frequency of the driving force $F(t)$, $\tilde{L}_{coh}(\Omega)$ is the corresponding Fourier coefficient in the Fourier series expansion of $L_{coh}(\tau)$, and $\tilde{L}_{incoh}(\Omega)$ is the Fourier transform at frequency Ω of $L_{incoh}(\tau)$. Taking into account Eq. (10), it follows immediately that R_{out} is enhanced with respect to that of an individual subunit; namely,

$$R_{out} = NR_{out}|_{indiv}. \tag{12}$$

Similarly, the collective input signal-to-noise ratio $R_{in}|_{coll}$ is defined by replacing the numerator and denominator in Eq. (11) with the corresponding quantities for the collective input $F(t)+(1/N)\Sigma_i \xi_i(t)$.

It is interesting to compare R_{out} with $R_{in}|_{coll}$. Then, we compute the stochastic resonance gain as

$$G = \frac{R_{out}}{R_{in}|_{coll}}. \tag{13}$$

In linear systems, the value of the output signal-to-noise ratio is identical to that of the input, and the gain is just 1. For nonlinear systems this is no longer the case. Taking into account that

$$R_{in}|_{coll} = NR_{in}|_{indiv}, \tag{14}$$

we see that the gain associated with the collective variable will be the same as the one associated to a single unit; namely,

$$G|_{coll} = G|_{indiv}. \tag{15}$$

Let us now consider a set of N *interacting identical subsystems*. The introduction of interaction terms influences the behavior of each subunit, and, therefore, that of the global variable $S(t;\theta)$. To indicate that we are dealing with a coupled set, we will use the notation $(t;\theta)$, where θ is a parameter gauging the strength of the coupling term. Even in the presence of coupling, as the subunits are still identical, we can still write

$$\langle S(t;\theta)\rangle = \langle x_1(t;\theta)\rangle. \tag{16}$$

For the second cumulant we obtain

$$\langle S^2(t;\theta)\rangle - \langle S(t;\theta)\rangle^2 = \frac{1}{N}\langle x_1^2(t;\theta)\rangle - \langle x_1(t;\theta)\rangle^2 + \left(1-\frac{1}{N}\right)\langle \delta x_1(t,\theta)\,\delta x_2(t,\theta)\rangle, \tag{17}$$

where we have separated the individual variables as $x_j(t;\theta) = \langle x_j(t;\theta)\rangle + \delta x_j(t;\theta)$. The two-time correlation function is

$$\langle S(t;\theta)S(t+\tau;\theta)\rangle = \frac{1}{N}\langle x_1(t;\theta)x_1(t+\tau;\theta)\rangle + \left(1-\frac{1}{N}\right)\langle x_1(t;\theta)x_2(t+\tau;\theta)\rangle. \tag{18}$$

In the long t limit, the two-time correlation functions are periodic in t with period T. Thus, after cycle averaging over t we write

$$L(\tau;\theta) = \frac{1}{N}C(\tau;\theta) + \left(\frac{N-1}{N}\right)C^{(12)}(\tau;\theta), \tag{19}$$

where

$$C(\tau;\theta) = \frac{1}{T}\int_0^T dt\langle x_1(t;\theta)x_1(t+\tau;\theta)\rangle_* \tag{20}$$

and

$$C^{(12)}(\tau;\theta) = \frac{1}{T}\int_0^T dt\langle x_1(t;\theta)x_2(t+\tau;\theta)\rangle_*. \tag{21}$$

The two-time cross correlation function can be split as

$$\langle x_1(t;\theta)x_2(t+\tau;\theta)\rangle_* = \langle x_1(t;\theta)\rangle_*\langle x_2(t+\tau;\theta)\rangle_* + \langle \delta x_1(t;\theta)\,\delta x_2(t+\tau;\theta)\rangle_*.$$

Separating $C(\tau;\theta)$ and $C^{(12)}(\tau;\theta)$ into their coherent and incoherent parts, and taking into account that the coherent part of the cross correlation function is identical to $C_{coh}(\tau;\theta)$, we have

$$\frac{1}{N}C(\tau;\theta) + \left(\frac{N-1}{N}\right)C^{(12)}(\tau;\theta) = \frac{1}{N}C_{coh}(\tau;\theta) + \frac{1}{N}C_{incoh}(\tau;\theta) + \left(\frac{N-1}{N}\right)C_{coh}(\tau;\theta) + \left(\frac{N-1}{N}\right)C_{incoh}^{(12)}(\tau;\theta) \tag{22}$$

$$= C_{coh}(\tau;\theta) + \frac{1}{N}C_{incoh}(\tau;\theta) + \left(\frac{N-1}{N}\right)C_{incoh}^{(12)}(\tau;\theta), \tag{23}$$

which shows that the coherent part of the collective correlation function is just the coherent part of one of the individual subsystems, but the incoherent part is affected by the cross correlation term. We can then conclude that, in the presence of interactions,

$$L_{coh}(\tau;\theta) = C_{coh}(\tau;\theta), \quad L_{incoh}(\tau;\theta) = \frac{1}{N}C_{incoh}(\tau;\theta) + \left(\frac{N-1}{N}\right)C_{incoh}^{(12)}(\tau;\theta). \tag{24}$$

It is clear that, for $N \gg 1$, the incoherent part of the cross correlation term will dominate the incoherent part of the global correlation function. $L_{coh}(\tau;\theta)$ is periodic in τ with the period of the driver, while $L_{incoh}(\tau;\theta)$ decays to zero for long τ. The signal-to-noise ratio of the collective variable, $R_{out}(\theta)$ can then be evaluated by replacing $\tilde{L}_{coh}(\Omega)$ and $\tilde{L}_{incoh}(\Omega)$ by $\tilde{L}_{coh}(\Omega;\theta)$ and $\tilde{L}_{incoh}(\Omega;\theta)$ in Eq. (11). Notice that in the presence of coupling, the relation between the signal-to-noise ratio of the collective variable is not related to that of a single subunit by a simple relation as in Eq. (12). Also, the gain $G(\theta)=R_{out}(\theta)/R_{in}|_{collec}$ does not have to coincide with that of a single subsystem.

III. NUMERICAL RESULTS

In general, nonlinearities preclude analytical solutions of Eq. (1). We will then use numerical simulations to obtain useful information about the stochastic process $S(t)$. Clearly, the specific details will depend upon the type of coupling between the subunits. In this paper we will consider a model introduced by Desai and Zwanzig [9] formed by a set of noisy bistable subunits with mean-field coupling; namely,

$$U(x_1, \ldots ,x_N) = \sum_{i=1}^{N} V(x_i) + \frac{\theta}{2}\sum_{i=1}^{N} x_i^2 - \frac{\theta}{2N}\left(\sum_{i=1}^{N} x_i\right)^2, \quad (25)$$

with

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}. \quad (26)$$

In the asymptotic limit $N\to\infty$, Desai and Zwanzig showed that the statistical properties of the model could be analyzed in terms of a nonlinear Fokker-Planck equation which allows the coexistence of several stable probability distributions for some ranges of noise strengths and $\theta<1$. In the same asymptotic limit, we analyzed a few years ago the stochastic resonant behavior of the first moment, $\langle S(t)\rangle_*$, when the system is driven by a time dependent sinusoidal force, using a combination of analytical and numerical procedures [10]. In particular, for very weak input amplitudes, a linear response theory analysis showed that a huge amplification in the amplitude of the average output $\langle S(t)\rangle_*$ with respect to that of the driving force could be achieved.

In this work we concentrate on situations where (i) the number of subunits is finite; (ii) the amplitude of the driving force is not necessarily small and a linear response theory approximation is, in principle, ruled out. Furthermore, we are interested not only on the average behavior of the global variable, but more importantly, on the behavior of the fluctuations. Thus, our goal is to analyze numerically the long time behavior of the first two cumulant moments of the collective process and its autocorrelation function, for coupled and uncoupled finite sets of identical units. In the following, we present our results obtained with two types of external driving: a sinusoidal force $F(t)=A\sin(\Omega t)$ and a periodic rectangular input defined as

$$F(t) = (-1)^{n(t)}A, \quad (27)$$

where $n(t)=\lfloor 2t/T\rfloor$, $\lfloor z\rfloor$ being the floor function of z, i.e., the greatest integer less than or equal to z. In other words, $F(t)=A[F(t)=-A]$ if $t\in[nT/2,(n+1)T/2]$ with n even (odd).

The numerical method used to solve the Langevin dynamics has been detailed in our previous work [16]. The integration algorithm is based on one of the schemes put forward by Greenside and Helfand [17]. The algorithm involves four stages and it is of order between h^3 and h^4 in the integration time step h. This is very advantageous as it allows us to use relatively large time steps to speed up the simulations. We also notice that our averages are always averages over many realizations of the noise $\xi(t)$ (5000 for the results below). Once the numerical approximations to $L(\tau;\theta)$ and $L_{coh}(\tau;\theta)$ have been evaluated, it is a matter of carrying out numerical quadratures at the desired frequency to calculate the signal-to-noise ratio and the gain [see Eqs. (11) and (13)].

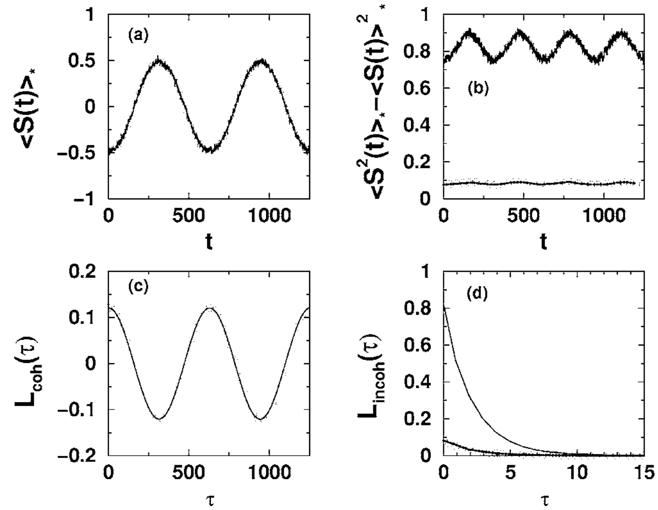

FIG. 1. Temporal behavior of the first two cumulants [(a) and (b), respectively], and the coherent and incoherent parts of the correlation function of the collective variable [(c) and (d), respectively] for a system of N uncoupled subsystems. Parameter values are $D=0.52$, $A=0.3$, and $\Omega=0.01$. Solid lines correspond to $N=1$ and dotted lines to $N=10$. Circles correspond to the values for $N=1$ divided by 10.

Let us first consider the case of independent subunits, $\theta=0$. In Fig. 1 we display the time behavior of the first two cumulant moments $\langle S(t)\rangle_*$ (a) and $\langle S^2(t)\rangle_*-\langle S(t)\rangle_*^2$ (b), and the two contributions to the correlation function, $L_{coh}(\tau)$ (c) and $L_{incoh}(\tau)$ (d), for a single unit, $N=1$, and for a set of $N=10$ noninteracting identical subunits. The driving force is sinusoidal with amplitude $A=0.3$ and frequency $\Omega=0.01$. The noise strength is $D=0.52$. This is a relatively large value of the noise strength in relation to the noise value at which the signal-to-noise ratio shows its peak (see the upper panel in Fig. 2). As expected from the exact results discussed in Sec. II, the time behaviors of $\langle S(t)\rangle_*$ and $L_{coh}(\tau)$ obtained from the numerics are independent of the system size [the two curves in panels (a) and (c) of Fig. 1 coincide]. The expected strong reduction of the fluctuations as the system size is increased [see Eqs. (6) and (10)] can be clearly observed in the behaviors of the second cumulants and the incoherent part of the correlation functions.We have also compared $[\langle x_1^2(t)\rangle_*-\langle x_1(t)\rangle_*^2]/10$ and $C_{incoh}(\tau)/10$ [circles in panels (b) and (d) in Fig. 1] with $\langle S^2(t)\rangle_*-\langle S(t)\rangle_*^2$ and $L_{incoh}(\tau)$ and, as seen in the figures, the agreement is very good. The dependence of the signal-to-noise ratio and the gain with D for the same driving parameters are depicted in Fig. 2. To evaluate the gain, we have used that for the sinusoidal driving

$$R_{in}\Big|_{indiv} = \frac{\pi A^2}{4D}, \quad (28)$$

while for the rectangular input

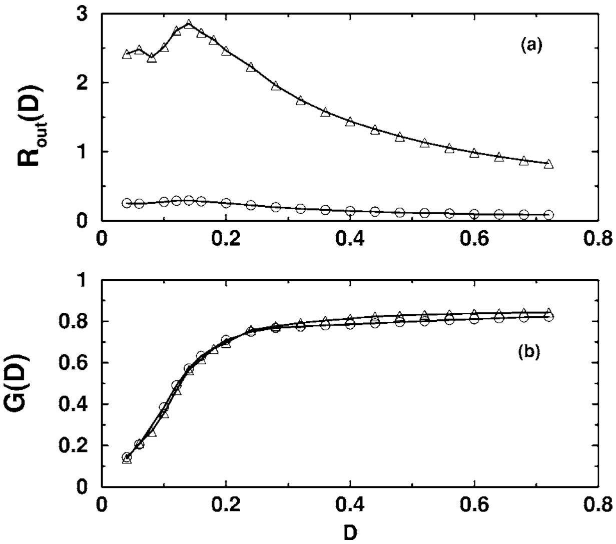

FIG. 2. Dependence of the signal-to-noise ratio R_{out} (a) and the gain (b), for a system of N independent subsystems. Parameter values are the same as those in Fig. 1. Open circles correspond to $N=1$ and triangles to $N=10$. The lines have been drawn as a guide to the eye.

$$R_{in}\Big|_{indiv} = \frac{4A^2}{\pi D}. \tag{29}$$

The enhancement of R_{out}, proportional to the system size, is as expected according to Eq. (12). Also, the gain is independent of the system size [Eq. (13)]. Notice that for the case of subthreshold sinusoidal driving forces considered in Fig. 2, the gain values are always below 1, even though the signal-to-noise ratio reaches rather high values for $N=10$. Later on, we will see that, for coupled systems, gains larger than 1 can be obtained with subthreshold sinusoidal inputs.

We will now consider finite sets of coupled subunits ($\theta>0$) and analyze the influence of the coupling strength and the system size on the collective response of the system. Let us first study the influence of the strength of the coupling parameter θ, on the system response when the driving parameters and the number of subsystems are kept fixed. In Fig. 3 we plot the time behavior of the first two cumulants of the global variable, $\langle S(t;\theta)\rangle_*$ and $\langle S^2(t;\theta)\rangle_* - \langle S(t;\theta)\rangle_*^2$, as well as the coherent and incoherent contributions of the correlation function for $N=10$. The external driving force is sinusoidal with a frequency $\Omega=0.01$ and amplitude $A=0.3$. The noise strength is kept at $D=0.52$ as in Fig. 1. The noise value has been chosen so that it roughly corresponds to the noise value at which $R_{out}(\theta)$ reaches its maximum for $\theta=1.5$ (see the upper panel in Fig. 4). Two values of the coupling parameter have been used, $\theta=0.5$ (solid line) and 1.5 (dotted line). As seen in (a), the average output is periodic in time with the period of the external driving. Even though the driving force is sinusoidal, the shape of the average response shows a certain degree of distortion with respect to a pure monochromatic signal, due to the generation of higher harmonics. Comparison with the results in Fig. 1(a) indicates that, at this rather large noise value, coupling between subunits enhances the average output with respect to

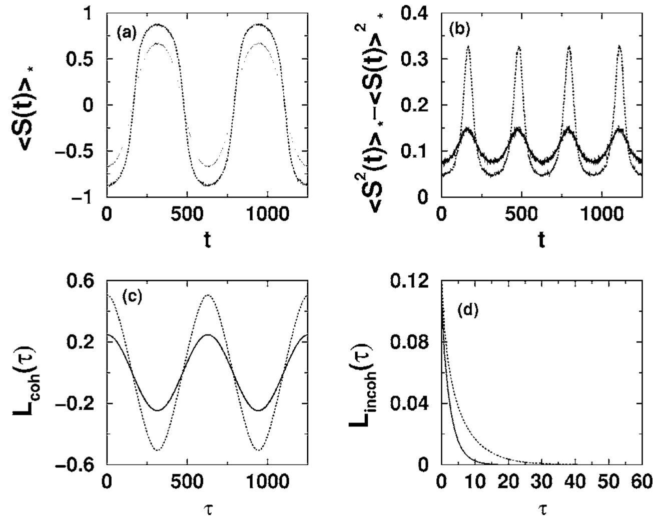

FIG. 3. Temporal behavior of the first two cumulants [(a) and (b), respectively], and the coherent and incoherent parts of the correlation function of the collective variable [(c) and (d), respectively] for a system of $N=10$ identical subsystems with mean-field coupling. In all panels, the solid lines correspond to $\theta=0.5$ and the dotted lines to $\theta=1.5$. The rest of the parameter values are $D=0.52$, $A=0.3$, and $\Omega=0.01$.

that existing in the absence of coupling. The influence of the coupling is more dramatic on the behavior of the second cumulant, as depicted in Fig. 3(b). Comparison with the results presented in Fig. 1(b) clearly shows the influence of the cross terms brought about by the interactions between the subunits [see Eq. (17)]. The second cumulant has a time-periodic behavior with a period half the period of the driving force. It reaches its maximum value at those instants of time when the first moment is near zero. Even though the level of

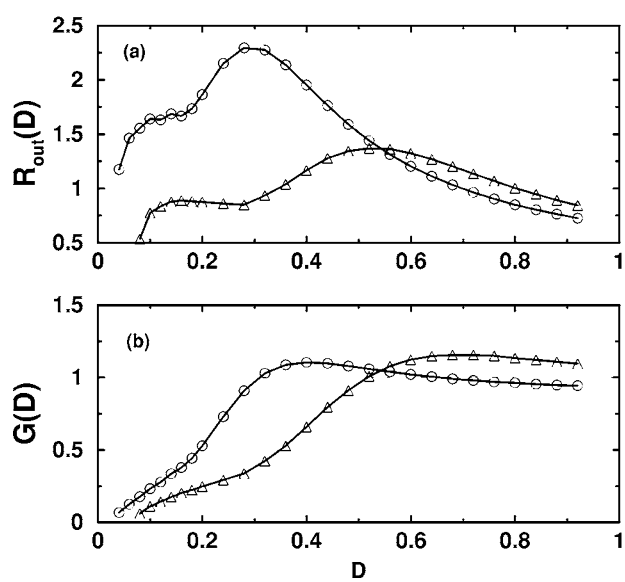

FIG. 4. Dependence of the signal-to-noise ratio R_{out} (panel (a)) and the gain (panel (b)), for a set of $N=10$ identical subsystems with mean-field coupling. Open circles correspond to $\theta=0.5$ and triangles to $\theta=1.5$. The rest of the parameter values are the same as in Fig. 3.

the collective fluctuations for coupled sets is higher than those existing in an uncoupled one, the fluctuations are still significantly smaller than those arising in the response of a single system. The time behaviors of the coherent part of the correlation functions in Fig. 3(c) are consistent with those of the first moment. As shown in Fig. 3(d) the incoherent parts of the correlation functions decay relatively fast from rather small initial values (which are the cycle averages of the corresponding second cumulant).

The signal-to-noise ratio and the gain as functions of D can readily be evaluated. For a sinusoidal driver with the same parameter values as in Fig. 3, we have obtained the results for $R_{out}(\theta)$ (a) and $G(\theta)$ (b) displayed in Fig. 4, for $\theta=0.5$ and 1.5. The signal-to-noise ratio shows the nonmonotonic behavior with D typical of stochastic resonance. Taking into account the behavior of R_{out} when $\theta=0$ (see Fig. 2), one can conclude that coupling shifts the maximum of $R_{out}(\theta)$ to higher values of D as θ increases. This feature in the response of coupled sets opens up an interesting possibility which does not exist when dealing with a single unit. For a given value of the noise strength, the coupling parameter could be adjusted accordingly so $R_{out}(\theta)$ is maximized. Another interesting consequence of coupling is that, even though the peaks of $R_{out}(\theta)$ are reduced with respect to those when $\theta=0$, the gain in coupled sets can be larger than unity. In principle, nothing precludes gains larger than 1 away from a linear response regime. But we are not aware of any reported data showing $G>1$ for a single system driven by a sinusoidal signal with subthreshold amplitude. Gains larger than 1 in noisy bistable systems driven by subthreshold inputs seem to require multifrequency driving forces [16,18]. The reason why $G(\theta)$ can be larger than 1 in coupled sets is related to the shift of the $R_{out}(\theta)$ peak location to higher values of D as θ increases. At those high noise values, R_{in} is large [see Eq. (28)], but $R_{out}(\theta=0)$ is significantly smaller than $R_{out}(\theta>0)$, and consequently, the ratio defining the gain [Eq. (13)] can achieve higher values for couple sets than for noninteracting ones.

In Figs. 5 and 6 we depict the results obtained for the same quantities and parameter values as in Figs. 3 and 4, but for a rectangular input with $T=2\pi/\Omega$. As we have shown in our previous work on SR in single units [2,16], the fluctuations at the output with multifrequency input forces are very much reduced compared with those present at the output when the input force is sinusoidal. This strong reduction gave rise to the possibility of observing gains larger than 1 for subthreshold multifrequency inputs in single noisy bistable systems. An analogous situation exists for arrays of coupled systems. The signal-to-noise ratio and the gain of the collective variable of a finite set of coupled units driven by a rectangular signal are very much enhanced compared with those obtained with a sinusoidal driving with the same amplitude and period. Notice that now $R_{out}(\theta)$ has only a single peak, whose location shifts to larger values of the noise as θ increases.

We now turn our analysis to the influence of the number of subsystems on $S(t)$. As discussed in Sec. II, for uncoupled sets, the effect of the system size is to reduce by a factor $1/N$ the size of the fluctuations associated to a single subunit.

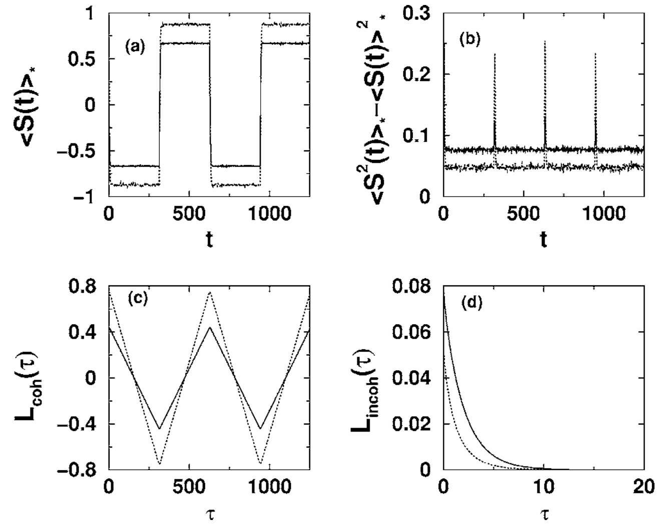

FIG. 5. Same as Fig. 3 for the rectangular input given by Eq. (27) with $\Omega=0.01$ and $A=0.3$.

When coupling exists, this is no longer the case as the behavior of the fluctuations of $S(t)$ is also tied to that of the cross correlation of two coupled subunits.

We consider (see Fig. 7) the response to a sinusoidal driving force with amplitude $A=0.3$ and frequency $\Omega=0.01$. The noise strength is $D=0.52$. We have used $\theta=1.5$ and $N=5,10,20$. The first cumulant (a) and $L_{coh}(\tau;\theta)$ (c) do not show a strong dependence on the number of subunits. On the other hand, the fluctuations decrease drastically as N increases. For large systems, the second cumulant (b) is basically different from zero only during a relatively small time interval within each period. The second cumulant behavior for $\theta>0$ differs from its behavior for uncoupled sets, due to the existence of new harmonics in the output and the influence of the cross second cumulant $\langle \delta x_1(t;\theta)\delta x_2(t+\tau;\theta)\rangle$ [see Eq. (17)]. Consistently with the behavior of the second cu-

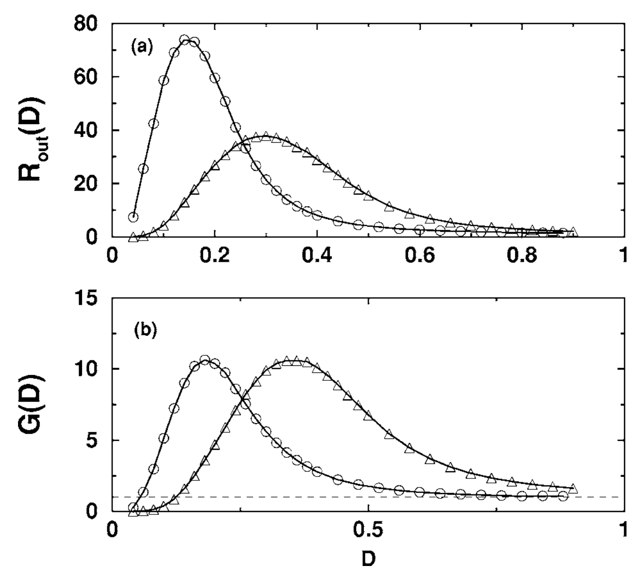

FIG. 6. Same as Fig. 4 for the rectangular input given by Eq. (27) with $\Omega=0.01$ and $A=0.3$.

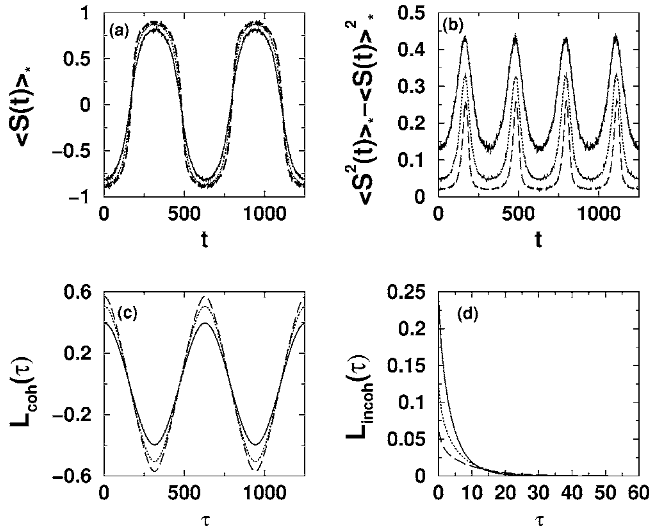

FIG. 7. Time behavior of the first two cumulants (a) and (b) and of the coherent and incoherent parts of the correlation function (c) and (d) of the variable $S(t)$ for sets of different sizes. The driving force is sinusoidal with $A=0.3$ and $\Omega=0.01$. Solid lines correspond to $N=5$, dotted lines to $N=10$, and dashed lines to $N=20$. The coupling is $\theta=1.5$.

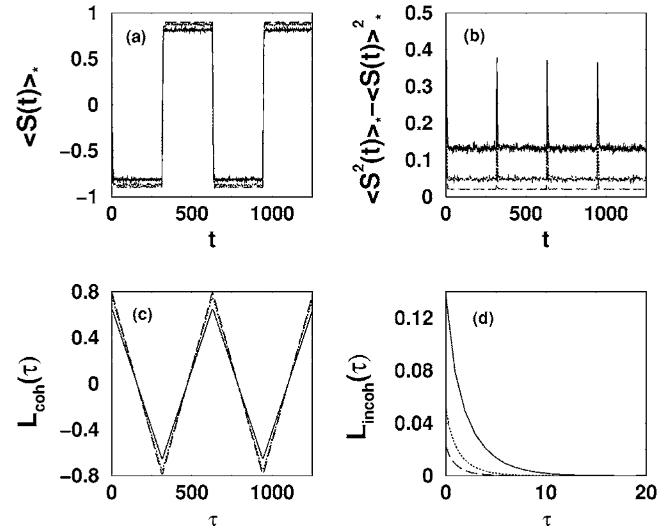

FIG. 9. Same as Fig. 7 for the rectangular input given by Eq. (27) with $\Omega=0.01$ and $A=0.3$.

mulant as N increases, the incoherent part of the correlation function gets smaller as the system increases its size. The behavior of the signal-to-noise ratio [Fig. 8(a)] and the gain [Fig. 8(b)] with respect to D for systems with different number of subsystems ($N=5,10,20$), driven by a sinusoidal force with $\Omega=0.01$ and amplitude $A=0.3$ is displayed in Fig. 8. The coupling parameter θ is kept fixed at 1.5. The three plots for the signal-to-noise ratio for the three values of N present two broad maxima at two different values of the noise strength. The location of the maxima shifts to higher values of D as the system size is increased. If the size of the system could be adjusted, one could do so accordingly with the existing noise level to get a maximized output. Gains larger than unity can also be obtained as N increases. The range of D values for which $G>1$ depends on N. For rectangular driving forces the different quantities show the same qualitative behavior for the different sizes as in the case of sinusoidal inputs (see Figs. 9 and 10), but SR effects with multifrequency inputs are pretty much enhanced with respect to the ones obtained with sinusoidal driving terms.

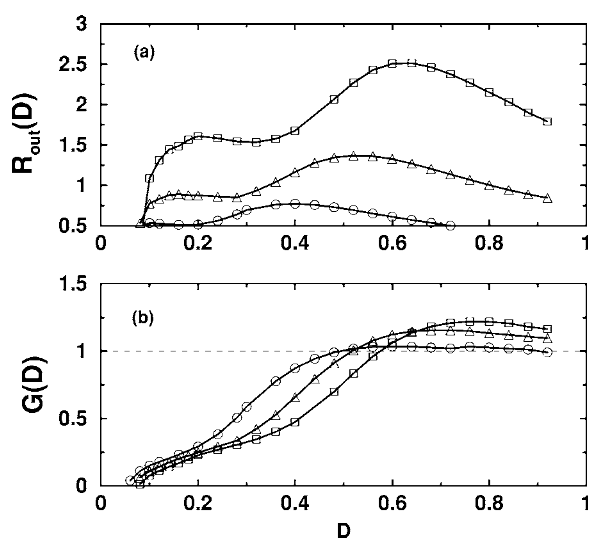

FIG. 8. Dependence on D of the signal-to-noise ratio and the gain for the same situations as in the previous figure. Circles correspond to $N=5$, triangles to $N=10$, and squares to $N=20$. Solid lines have been drawn as a guide to the eye.

IV. CONCLUSIONS

In this work we have explored stochastic resonance effects in finite sets of coupled, identical, noisy units driven by time-periodic forces. The whole set is characterized by a collective stochastic variable, which we consider to be the arithmetic mean of the random variable describing each sub-

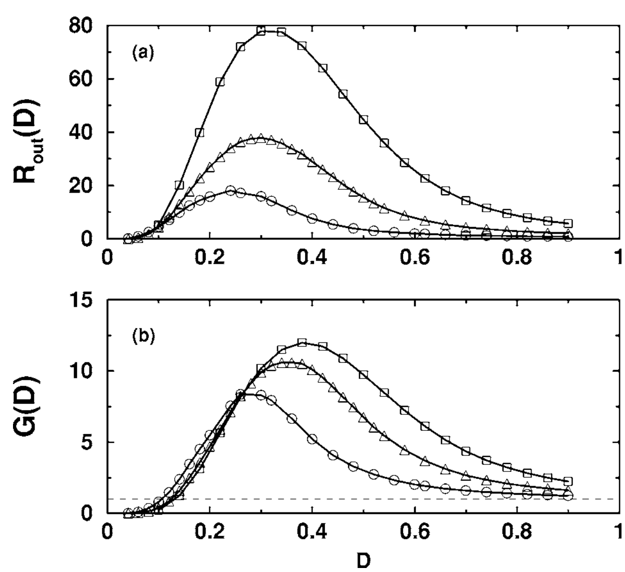

FIG. 10. Same as Fig. 8 for the rectangular input given by Eq. (27) with $\Omega=0.01$ and $A=0.3$.

unit. We have focused on the first two cumulant moments and the one-time correlation function of the global variable. The lack of analytical solutions for the set of stochastic differential equations describing the dynamical evolution leads us to use numerical procedures. Averaging over many realizations of the noise, we estimate the time evolution of the cumulants and the correlation function for sets of varying sizes and for different values of their coupling terms. We find that the collective fluctuations are pretty much reduced with respect to those present in single isolated units. The amount of reduction depends on the system size and the coupling strength.

Our results indicate that the collective variable shows stochastic resonant effects regardless of the strength of the coupling or the size of the system. Furthermore, the peak values of the several SR quantifiers of the collective variable are enhanced with respect to those obtained for each independent subunit. The main reason for this enhancement lies in the tremendous reduction and control of the output fluctuations brought up by the interaction among the different subunits. As we have explicitly demonstrated, coupling together a modest number of individual systems, opens the possibility of obtaining a collective output which is stronger and a lot less noisy than the output of each of them taken individually.

If the signal-to-noise and the gain are taken as indications of the "quality" of the output signal with respect to the input, we can conclude that increasing the system size and increasing the value of the coupling term seems to be a reliable way to get an enhancement of the quality of the output signal with respect to the output of either a single unit or sets of uncoupled subunits.

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